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GRAPHICAL FOURIER OPERATIONS.

UNITED STATES
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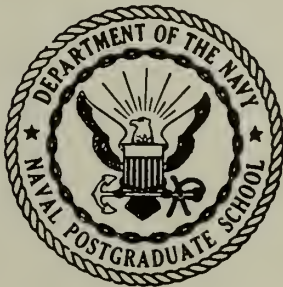
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by

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ABSTRACT: The graphical, symbolic function method of evaluating the discrete or continuous Fourier spectrum of a function is proved by use of symbolic analysis.

After a review of pertinent symbolic function theory, the method is illustrated for several problems including the non-finite polynomial function case.

There exists a technique for finding Fourier Series coefficients which generally is shorter than the usual integration technique. It is the purpose of this report to show the most salient points of the method.

We shall first have to introduce a few concepts from the theory of distributions.

Let us consider operator functions $s(t)$, which operate inside integrals on appropriate object functions $\phi(t)$ to produce functionals $F_s(\phi)$,

$$F_s(\phi) = \int_a^b s(\xi)\phi(\xi)\,d\xi.$$

Clearly, if we change $\phi(t)$ for fixed $s(t)$, we will generate a mapping of the function space $\{\phi(t)\}$ into the space $F_s\{\phi\}$; further if $\phi(t)$ is continuous, analytic, etc., and $s(t)$ is integrable, a continuous, linear functional $F_s(\phi)$ will result.

Suppose, alternately, we consider the class of functions $\{\phi(t)\}$ known, <u>define</u> $F_s(\phi)$ and ask what is $s(t)$? It soon becomes clear that $s(t)$ will not necessarily be analytic, in fact it may be impossible to

define an integrable function having the desired properties for s(t). Never-theless, several such functions have proved to be of great usefulness. Examples of such functions in common use are

1.   Heaviside unit step function, $Us(t)$

2.   Dirac unit impulse or delta function, $\delta(t)$

3.   The signed function, $sgn(t)$

4.   The square function $sq(t)$

5.   The convergence function $cv_\sigma(t)$

These functions are called symbolic functions. The obvious questions are:

1.   How are such functions to be defined and interpreted?

2.   What properties do they have?

3.   How may they be applied to problems?

We shall consider 1, 2, and the title of this report provides the application 3.

<u>Def</u>: Let $\{\phi(t)\}$ be the class of functions of all continuous functions with the properties

$$d^n\phi/dt^n \text{ exists,}$$

$\phi'(t)$ and all higher derivatives vanish at least as fast as

$1/|t|$ at ∞ ,

$$\int_{-\infty}^{\infty} |\phi(t)|^2 dt < \infty$$

Then the symbolic function s(t) is the linear mapping function, which maps the space $\{\phi(t)\}$ into the functional space $F_s\{\phi(t)\}$ by means of inte-gration.

It will turn out that in many cases the class of functions $\{\phi(t)\}$ we have defined as the object space is too restrictive, i.e., our conditions on $\phi(t)$ are sufficient but they may in some cases be more than necessary.

In general, each exceptional case will have to be separately examined.

Let us now give the defining integrals for the various functions given as examples.

1. The Heaviside unit step function

$$Us(t) \implies \int_{-\infty}^{\infty} Us(t) \phi'(t) \, dt = \int_{0}^{\infty} \phi(t) \, dt$$

This is equivalent to an ordinary piecewise continuous function with values:

$$Us(t) = 0 \quad , \quad t < 0$$
$$= 1/2 \quad , \quad t = 0$$
$$= 1 \quad , \quad t > 0$$

2. The delta function $\delta(t)$

$$\delta(t) \implies \int_{-\infty}^{\infty} \delta(t) \phi(t) \, dt = \phi(0)$$

This function cannot be defined in the ordinary sense; however it behaves heuristically like:

$$\delta(t) = 0 \ , \quad t \neq 0$$
$$= \infty \ , \quad t = 0$$

such that $\int_{-\infty}^{\infty} \delta(t) \, dt = 1$.

3. The signed function

$$sgn(t) \implies \int_{-\infty}^{\infty} sgn(t) \phi(t) \, dt = \int_{0}^{\infty} \phi(t) \, dt - \int_{-\infty}^{0} \phi(t) \, dt$$

It is equivalent to the ordinary function

$$sgn(t) = -1 \ , \quad t < 0$$
$$= 0 \ , \quad t = 0$$
$$= +1 \ , \quad t > 0.$$

4. The square or box car function

$$sq_T(t) \implies \int_{-\infty}^{\infty} sq_T(t) \phi(t) \, dt = \int_{0}^{T} \phi(t) \, dt$$

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It is equivalent to the ordinary function

$$sq_T(t) = \quad 0 \quad , \quad t < 0$$

$$= 1/2 \quad , \quad t = 0, T$$

$$= 1 \quad , \quad 0 < t < T$$

$$= 0 \quad , \quad t > T$$

5. The convergence function $cv_\alpha(t)$

$$cv_\alpha(t) \quad \Rightarrow \quad \int_{-\infty}^{\infty} cv_\alpha(t)\phi(t)\,dt = \int_{-\infty}^{\infty} e^{-\alpha|t|}\phi(t)\,dt$$

so that

$$cv_\alpha(t) = e^{-\alpha|t|} \,, \quad (\alpha > 0)\,.$$

Note that we have taken the limits on the integrals as $(-\infty, \infty)$; in general we can talk about finite interval problems most easily by restricting the space of functions $\{\phi(t)\}$ to functions which vanish identically outside the interval of interest. In addition, it is easily seen that the above examples do not exhaust the possible symbolic functions; one could produce at least an indefinitely greater number just by multiplying a symbolic function by the class of continuous, integrable functions and defining each resulting function as a new symbolic function. A simple example which is fairly common is the ramp function

$$rp(t) = tUs(t) \quad \Rightarrow \quad \int_{-\infty}^{\infty} rp(t)\phi(t)\,dt = \int_{0}^{\infty} t\phi(t)\,dt$$
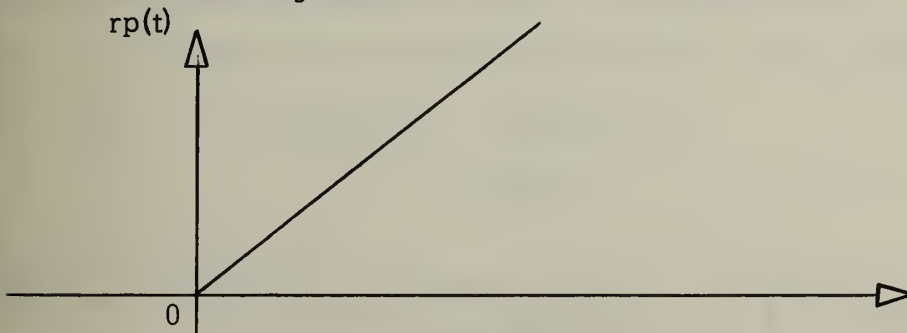
which is shown in Figure 1.

Fig. 1

Let us now consider what properties are possessed by symbolic functions. First, they are <u>linear</u> mappings. If $\phi_1(t)$ and $\phi_2(t)$ are any two functions belonging to $\{\phi(t)\}$ then for any symbolic function $s(t)$ such that

$$a\int_{-\infty}^{\infty} s(t)\phi_1(t)\,dt = aF_s(\phi_1)\,, \quad b\int_{-\infty}^{\infty} s(t)\phi_2(t)\,dt = bF_s(\phi_2)$$

then

$$F_s\left[a\phi_1+b\phi_2\right] = \int_{-\infty}^{\infty} s(t)\left[a\phi_1(t)+b\phi_2(t)\right]\,dt = \int_{-\infty}^{\infty} s(t)\,a\phi_1(t)\,dt + \int_{-\infty}^{\infty} s(t)\,b\phi_2(t)\,dt$$

$$= F_s(a\phi_1) + F_s(b\phi_2) = a\int_{-\infty}^{\infty} s(t)\phi_1(t)\,dt + b\int_{-\infty}^{\infty} s(t)\phi_2(t)\,dt$$

$$= aF_s(\phi_1) + bF_s(\phi_2)$$

<u>Differentiability</u>. Can any interpretation be given to $s'(t)$? In order to answer this two points are clear. We want $s(t)$ to satisfy as many of the usual properties of analysis as possible in order that it may have general applicability and, we must define $s'(t)$ by means of an integration process. The second point leads us to consider the integral

$$\int_{-\infty}^{\infty} s'(t)\phi(t)\,dt$$

and the first point suggests we try integration by parts. If integration by parts were possible, one would have

$$\int_{-\infty}^{\infty} s'(t)\phi(t)\,dt = \left. s(t)\phi(t)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} s(t)\phi'(t)\,dt$$

$$= -\int_{-\infty}^{\infty} s(t)\phi'(t)\,dt$$

since by hypothesis $\phi(t)$ vanishes as $|t| \to \infty$. Thus in order that ordinary integration by parts be valid, we define $s'(t)$ by:

<u>Def</u>: The derivative of a symbolic function $s(t)$ is the mapping $\phi(t) \xrightarrow{S'} -F_s(\phi')$

$$s'(t) \quad \Rightarrow \quad \int_{-\infty}^{\infty} s'(t)\phi(t)\,dt = -\int_{-\infty}^{\infty} s(t)\phi'(t)\,dt$$

$$= -F_s(\phi')\,.$$

In like manner we have

$$s^{(n)}(t) \quad \Rightarrow \quad \int_{-\infty}^{\infty} s^{(n)}(t)\phi(t)\,dt = \left. s^{(n-1)}(t)\phi(t)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} s^{(n-1)}(t)\phi'(t)\,dt$$

$$= -\int_{-\infty}^{\infty} s^{(n-1)}(t)\phi'(t)\,dt$$

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so that by induction

$$\int_{-\infty}^{\infty} s^{(n)}(t)\phi(t)\,dt = (-1)^n F_s(\phi^{(n)}).$$

Thus we take

Def: The nth derivative of a symbolic function $s(t)$ is the mapping

$$\phi(t) \xrightarrow{s^{(n)}} (-1)^n F_s(\phi^{(n)}).$$

To make this clear, let us take as an example

$$s(t) = \delta''(t), \quad \phi(t) = e^{-at},$$

then

$$\int_{-\infty}^{\infty} \delta''(t)e^{-at}\,dt = -\int_{-\infty}^{\infty}\delta'(t)\left[e^{-at}\right]'\,dt = (-1)(-1)\int_{-\infty}^{\infty}\delta(t)\left[e^{-at}\right]''\,dt$$

$$= F_\delta\left\{\,[\,e^{-at}\,]''\,\right\}$$

$$= (-a)^2 e^{-a(0)} = a^2$$

Note that when we apply this to the sgn (t) function, we have

$$\int_{-\infty}^{\infty}[\,sgn(t)\,]'\phi(t)\,dt = -\int_{-\infty}^{\infty}[\,sgn(t)\,]\phi'(t)\,dt = \int_{-\infty}^{0}\phi'(t)\,dt - \int_{0}^{\infty}\phi'(t)\,dt$$

$$= \phi(t)\Big|_{-\infty}^{0} - \phi(t)\Big|_{0}^{\infty} = 2\phi(0) = \int_{-\infty}^{\infty}[\,2\delta(t)\,]\phi(t)\,dt$$

so that operationally

$$[\,sgn(t)\,]' = 2\delta(t)$$

i.e., the two functions $[\,sgn(t)\,]'$ and $2\delta(t)$ represent the same mapping.

Def: Two symbolic functions $s_1(t)$, $s_2(t)$ are said to be equal if and only if they perform identical mappings on $\{\phi(t)\}$ i.e., if

$$\phi(t) \xrightarrow{s_1} F_{s_1}(\phi), \quad \phi(t) \xrightarrow{s_2} F_{s_2}(\phi) \text{ and if and only if } F_{s_1}(\phi) = F_{s_2}(\phi)$$

then

$$s_1(t) = s_2(t)$$

Another useful result is obtained by considering the derivative of the Heaviside function $Us(t)$

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$$\int_{-\infty}^{\infty} U\mathring{s}(t)\phi(t)\,dt = -\int_{-\infty}^{\infty} Us(t)\phi'(t)\,dt = -\int_{0}^{\infty} \phi'(t)\,dt = -\phi(t)\Big|_{0}^{\infty}$$

$$= \phi(0)$$

$$= \int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt$$

so that

$$U\mathring{s}(t) = \delta(t)$$

There are many other useful results which may be derived; some will occur in subsequent analyses. In particular we will mention a few more here which are almost obvious

$$1 = Us(t) + Us(-t)$$

$$\text{sgn}(t) = Us(t) - Us(-t)$$

$$\text{sg}_T(t) = Us(t) - Us(t-T)$$

We shall now prove a theorem which will be useful later.

<u>Theorem</u>: The derivative of a simply discontinuous function is the ordinary derivative where it exists plus impulse functions times the magnitude of jumps at the discontinuities.

<u>Proof</u>: A discontinuous function $f(t)$ with a jump of magnitude A at at $t = a$ may be represented by

$$f(t) = g(t) + A\,Us(t - a)$$

where $g(t)$ is a continuous function. Since $f(t)$ contains a symbolic function, it is also a symbolic function. Now one can easily prove that if $s_1(t)$ and $s_2(t)$ are symbolic functions and if

$$s_1(t) = s_2(t)$$

then

$$\mathring{s}_1(t) = \mathring{s}_2(t)$$

[Hint: Apply $s_1(t)$ and $s_2(t)$ to $\phi'(t)$]. Hence

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$$f'(t) = g'(t) + A\delta(t - a) \qquad\qquad Q.E.D.$$

Theorem: If

$$\phi(t) \overset{\delta}{\to} F_\delta(\phi) = \phi(o)$$

and $g(t)$ is a single valued function with a single valued inverse which vanishes for $t = t_o$

$$\delta(g(t)) = \frac{\delta(t - t_o)}{|g'(t_o)|}$$

or in general

$$\int_{-\infty}^{\infty} s(g(t))\phi(t)\,dt = \int_{g(-\infty)}^{g(\infty)} s(\xi)\phi(g^{-1}(\xi))\frac{d\xi}{g'(g^{-1}(\xi))}$$

$$= F_s \frac{\phi\{g^{-1}\}}{g'\{g^{-1}\}}$$

i.e.,

$$\phi(t) \overset{s(g)}{\to} F_s\left(\frac{\phi\{g^{-1}\}}{g'\{g^{-1}\}}\right)$$

Proof: Make change of variable.   Q.E.D.

Fourier expansions:

We are now in position to consider the application of symbolic functions to Fourier series. We note first of all that any periodic function can be written as a sum of appropriately placed pulses.

$$V(t) = \sum_{n=-\infty}^{\infty} g_n(t)$$

where

$$g_n(t) = g(t - nT)\Big[Us(t - nT) - Us(t - (n+1)T)\Big]$$

(See Fig. 2)

$g(t-nT)$

Fig. 2

nT $(n+1)T$ t
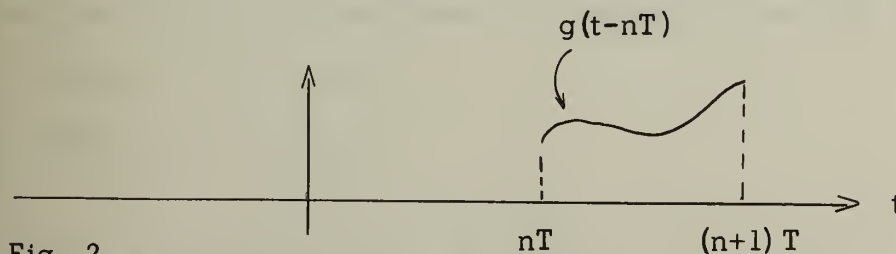
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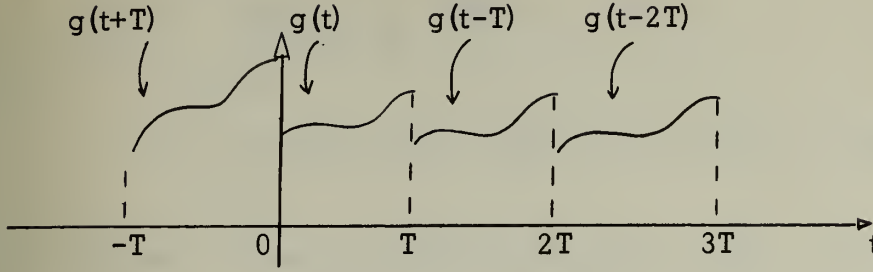Fig. 3 shows the sum of the terms $g_{-1}(t) + g_0(t) + g_1(t) + g_2(t)$

Fig. 3

Now if $V(t)$ is periodic, it must also have a Fourier series representation

$$V(t) = \sum_{k=-\infty}^{\infty} C_k e^{\frac{j2\pi nt}{T}}$$

where

$$C_k = \frac{1}{T} \int_0^T V(\xi) e^{-j\frac{2\pi k\xi}{T}} d\xi.$$

Hence we have

$$V(t) = \sum_{k=-\infty}^{\infty} C_k e^{\frac{j2\pi kt}{T}} = \sum_{n=-\infty}^{\infty} g(t-nT)\left[Us(t-nT) - Us(t-[n+1]T) \right]$$
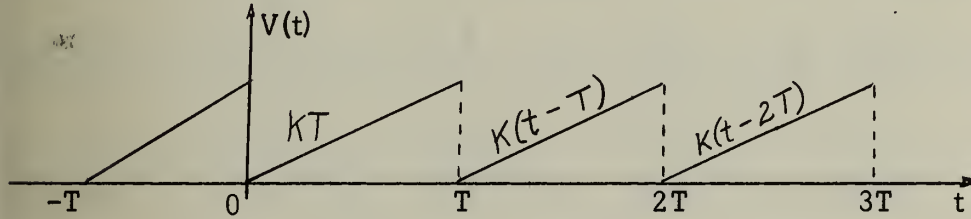

Fig. 4

Let us consider a typical waveform, the sawtooth wave (Fig. 4)

$$V(t) = Kt, \quad o < t < T$$

$$V(t) = V(t+T), \quad \text{all } t.$$

or

$$V(t) = \sum_{n=-\infty}^{\infty} K(t-nT\left[Us(t-nT) - Us(t-[n+1]T) \right]$$

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Suppose we differentiate the equation; we have

$$V'(t) = \sum_{k=-\infty}^{\infty} (\frac{j2\pi k}{T}) C_k e^{\frac{j2\pi k}{T} t}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ K[Us(t - nT) - Us(t - [n + 1] T)] - KT\delta(t - [n + 1] T) \right\}$$

Fig. 5 shows the derivative of the nth period , $(nT + \epsilon$ to $(n + 1)T + \epsilon)$
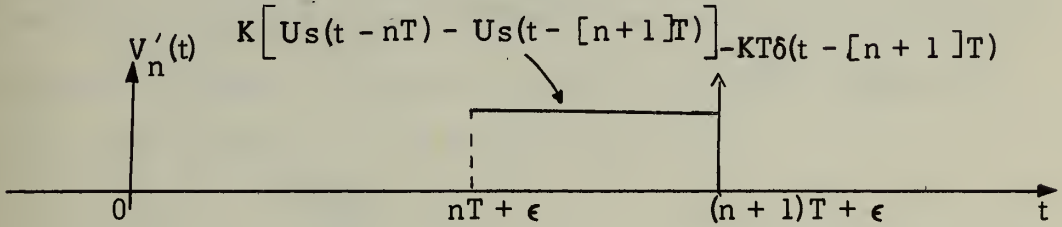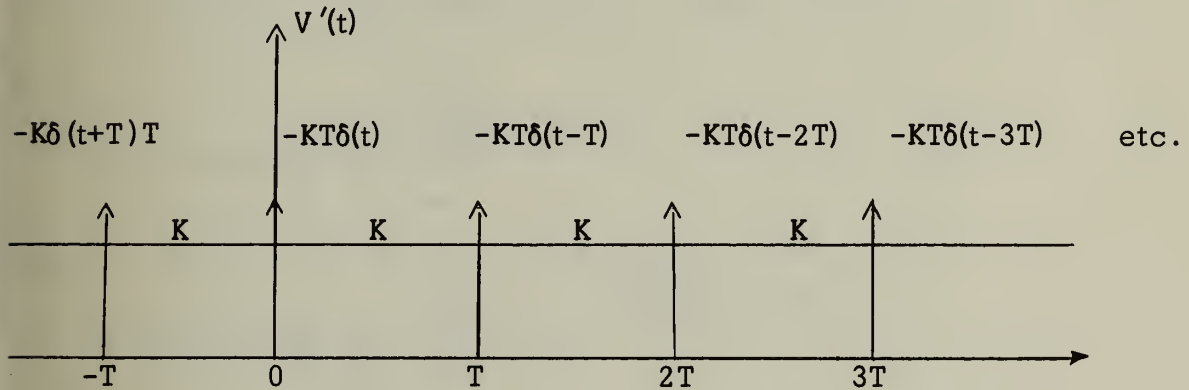
Fig. 5

and Fig. 6 the entire function

Fig. 6

If we differentiate the equation again, we have

$$V''(t) = \sum_{k=-\infty}^{\infty} C_k (\frac{j2\pi k}{T})^2 e^{\frac{j2\pi kt}{T}}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ K[\delta(t - nT) - \delta(t - [n + 1] T)] - KT\delta'(t - [n + 1] T) \right\}$$

Note (Fig. 7) that the resulting function $V''(t)$ now contains only impulse

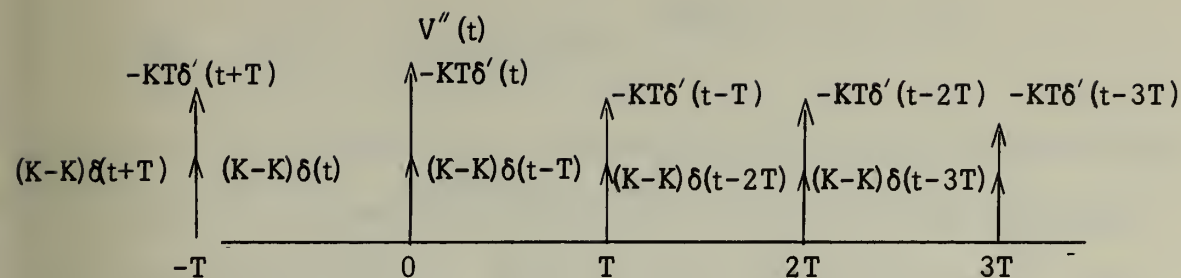functions and derivatives of impulse functions.

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Fig. 7

Let us see what the Fourier series representation is for such periodically spaced impulse functions. We have, by application of the coefficient formula to a delta function at $t = \alpha$, where $t < \alpha < t + T$

$$\delta_T(t - \alpha) = \sum_{m = -\infty}^{\infty} C_m e^{\frac{j 2\pi m t}{T}}$$

$$C_m = \frac{1}{T} \int_t^{t + T} \delta_T(t - \alpha) e^{\frac{-j 2\pi m t}{T}} dt = \frac{1}{T} e^{\frac{-j 2\pi m \alpha}{T}}$$

or

$$\delta_T(t - \alpha) = \frac{1}{T} \sum_{m = -\infty}^{\infty} e^{\frac{j 2\pi m}{T}(t - \alpha)}$$

Further

$$\delta_T'(t - \alpha) = \frac{1}{T} \sum_{m = -\infty}^{\infty} \left(\frac{j 2\pi m}{T}\right) e^{\frac{j 2\pi m}{T}(t - \alpha)}$$

these series representations should be committed to memory. Hence

$$V''(t) = \sum_{k = -\infty}^{\infty} \left(\frac{j 2\pi k}{T}\right)^2 C_k e^{\frac{j 2\pi k t}{T}} = \sum_{n = -\infty}^{\infty} \left\{ K \left[\frac{1}{T} \sum_{m = -\infty}^{\infty} e^{\frac{j 2\pi m}{T}(t - nT)} - \frac{1}{T} \sum_{m = -\infty}^{\infty} e^{\frac{j 2\pi m}{T}(t - [n + 1]T)} \right] \right.$$

$$\left. -KT \cdot \frac{1}{T} \sum_{m = -\infty}^{\infty} \left(\frac{j 2\pi m}{T}\right) e^{\frac{j 2\pi m}{T}(t - [n + 1]T)} \right\}$$

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$$= \sum_{m=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-K)\left(\frac{j2\pi m}{T}\right) e^{j\frac{2\pi m}{T}(t - [n+1]T)}$$

since the delta functions cancel. Now the coefficients of a Fourier series are the same for any period of the function, hence the above expression must hold independent of n. Let us take then the period for n = 0, i.e., from t = 0 to t = T, find C_k and then show that the result must hold when, say, n = ℓ. For the interval (0, T) we have for V"(t)

$$V''(t) = \sum_{k=-\infty}^{\infty} C_k \left(\frac{j2\pi k}{T}\right)^2 e^{j\frac{2\pi kt}{T}} = \sum_{m=-\infty}^{\infty} (-K)\left(\frac{j2\pi m}{T}\right) e^{j\frac{2\pi m}{T}(t - T)}$$

This expression must hold for arbitrary times within (0, T), thus can only be true if the coefficients of like exponential functions are equal. Hence setting m = k

$$C_k\left(\frac{j2\pi k}{T}\right)^2 = -K\left(\frac{j2\pi k}{T}\right) e^{\frac{-j2\pi kT}{T}}$$

$$= -K\left(\frac{j2\pi k}{T}\right)$$

or

$$C_k = \left(\frac{-K}{\frac{j2\pi k}{T}}\right)$$

Note if we had taken the ℓth interval of the function, the result is the same since

$$C_k = -\frac{K}{\left(\frac{j2\pi k}{T}\right)} e^{-j\frac{2\pi k}{T}(\ell + 1)T}$$

$$= \frac{-K}{\left(\frac{j2\pi k}{T}\right)}$$

We have considered here the complete analytic solution of the problem

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apart from the constant term which may be in the original $f(t)$. This will not be in our result because the constant disappears with differentiation. Note also that the coefficient result indicates this, somewhat, in that C_o does not exist. To obtain C_o we must do it in the conventional way

$$C_o = \frac{1}{T} \int_o^T f(\xi)\, d\xi = \frac{KT}{2}$$

We then write

$$V(t) = \frac{KT}{2} + \sum_{k=-\infty}^{\infty} -\frac{K}{j\frac{2\pi k}{T}} e^{j\frac{2\pi kt}{T}} (1 - \delta_{k,0})$$

where $\delta_{k,0}$ is the Kronecker delta

$$\delta_{k,m} = 0 \quad,\; k \neq m$$
$$\quad\;\; = 1 \quad,\; k = m$$

The result proven for this specific function can clearly be generalized to any periodic function.

Now in reveiwing our steps, we see that much time can be saved by doing the problem graphically. Before we do another example, let us outline the general method.

1. Draw several periods of $g(t)$, the periodic function for which the Fourier series is desired.

2. Graphically differentiate the function, putting in singularity functions at the discontinuities.

3. Continue to differentiate the function until the resulting function contains only delta functions or derivatives of delta functions (non-polynomial functions will be considered subsequently).

4. Write the appropriate Fourier series for the resulting derivative

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function by use of the series for a ℓth derivative of a delta function of magnitude A located at t = a for a fundamental period T.

$$A\delta_T^{(\ell)}(t - a) = \frac{A}{T} \sum_{k = -\infty}^{\infty} (\frac{j2\pi k}{T})^\ell e^{\frac{j2\pi k}{T}(t - a)} \qquad (\ell = 0, 1, 2, ...)$$

5. Suppose g(t) has been differentiated m times. C_k, the kth coefficient of the Fourier series for g(t) is obtained by dividing the coefficients of the series for $d^m g(t)/dt^m$ by $(\frac{j2\pi k}{T})^m$
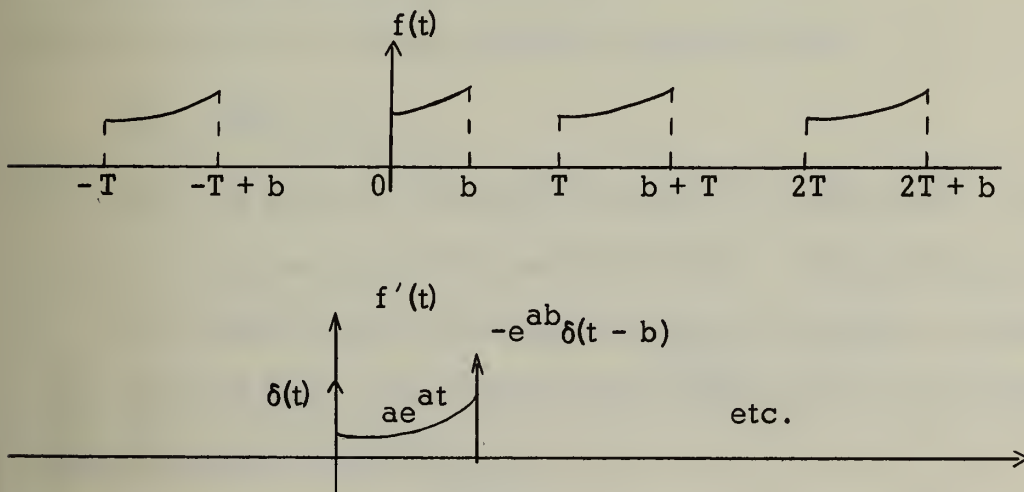
6. Obtain the d.c. term in the usual manner.

Example 2

$$f(t) = e^{at}, \quad o < t < b$$

$$= o \quad , \quad b < t < T.$$

$$f(t) = f(t + T)$$

We shall use the above procedure without comment and then consider the mental steps involved.

Note this is a case in which we will never get down to just delta functions. However, we see that in the period $(0^-, T^-)$

$$f'(t) = af(t) + \delta(t) - e^{ab}\delta(t - b)$$

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Now if

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{\frac{j2\pi kt}{T}}$$

then

$$f'(t) - af(t) = \sum_{k=-\infty}^{\infty} C_k \left[\left(\frac{j2\pi k}{T} \right) - a \right] e^{\frac{j2\pi kt}{T}}$$

Hence

$$C_k = \left[\frac{1}{T} - \frac{1}{T} e^{ab} e^{-j\frac{2\pi kb}{T}} \right] \Bigg/ \left[\left(\frac{j2\pi k}{T} \right) - a \right]$$

Mentally one follows somewhat the following process -- "I note f(t) is periodically differentiable, hence I differentiate until I get f(t) back again. In this case I see that

$$f'(t) = af(t) + \delta(t) - e^{ab} \delta(t - b)$$

or

$$f'(t) - af(t) = \delta(t) - e^{ab} \delta(t - b)$$

The left side has a Fourier coefficient representation

$$C_k \left[\frac{j2\pi k}{T} - a \right]$$

The right hand side has two delta functions both of period T. The first is at the origin and hence has no phase factor. Its coefficient is just $1/T$. The second delta function is of magnitude exp(ab), is negative in sign, and will have a phase factor exp $\left(\frac{-j2\pi k}{T} b \right)$ since it is located at b. Hence its Fourier coefficient is

$$\frac{1}{T} \cdot e^{ab} \cdot e^{-j\frac{2\pi kb}{T}}$$

The resulting series is thus

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{\left[\dfrac{1}{T} - \dfrac{1}{T}\, e^{ab}e^{\frac{-j2\pi kb}{T}}\right]}{\left[\dfrac{j2\pi k}{T} - a\right]} e^{\frac{j2\pi k}{T}t}$$

Do I need to put in the constant term? No, because in this case I have actually synthesized the original function completely, including the constant term. I also note in this case that, indeed, the value for k = 0 makes sense."
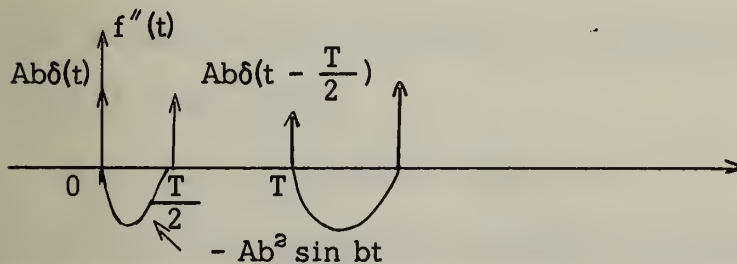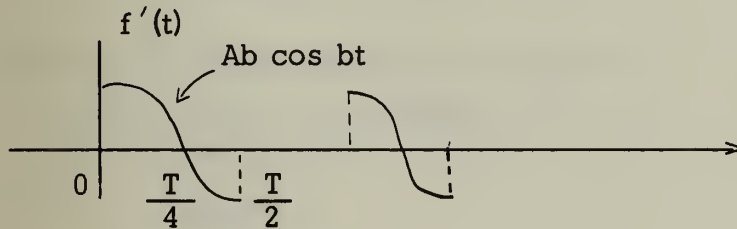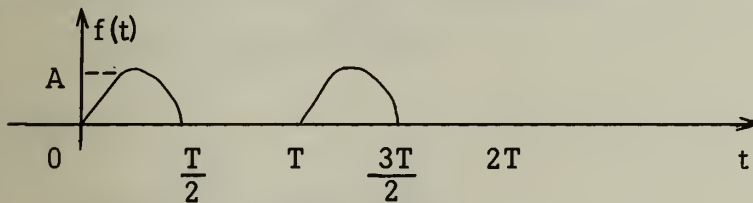
Example 3.

$$f(t) = A \sin bt \quad , \quad 0 < t < \frac{T}{2} \qquad\qquad bT = 2\pi$$

$$= 0 \qquad\qquad , \quad \frac{T}{2} < t < T$$

$$= f(t + T) \quad , \quad \text{all } t.$$

We shall do the work with no comment.

$$f''(t) = -b^2 f(t) + Ab\left[\delta(t) + \delta\left(t - \frac{T}{2}\right)\right]$$

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$$f''(t) + b^2 f(t) = Ab \left[\delta(t) + \delta\left(t - \frac{T}{2}\right) \right]$$

$$C_k = \frac{Ab \cdot \frac{1}{T} \left[1 + e^{-j\frac{2\pi k}{T} \cdot \frac{T}{2}} \right]}{\left(\frac{j2\pi k}{T}\right)^2 + b^2}$$

$$\therefore \ f(t) = \sum_{k=-\infty}^{\infty} \frac{Ab(1 + e^{-j\pi k})}{T\left[\left(\frac{j2\pi k}{T}\right)^2 + b^2 \right]} e^{\frac{j2\pi kt}{T}} \quad , \quad b = \frac{2\pi}{T}$$

The Fourier Integral.

One can also apply this method to the Fourier Integral. All that is required is to obtain an expansion for the delta function in the case of an infinite period. We have for the representation of any Fourier Integral – representable function

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$

where

$$G(f) = \int_{-\infty}^{\infty} g(\xi) e^{-j2\pi f\xi} d\xi$$

Hence for the delta function at a point $t = a$

$$G(f) = \int_{-\infty}^{\infty} \delta(t - a) e^{-j2\pi ft} dt$$
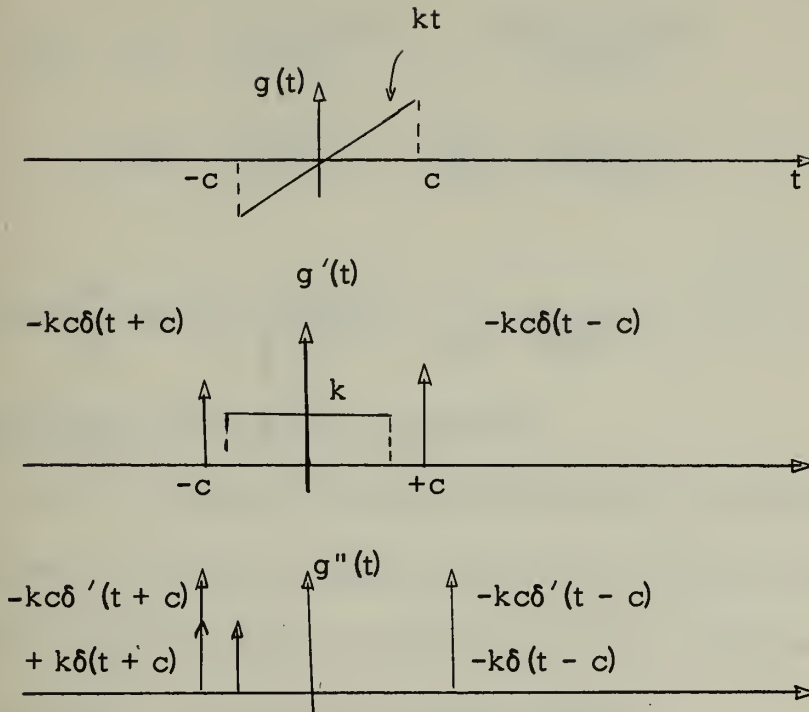
$$= e^{-j2\pi fa}$$

which is merely a delta function at $t = 0$ shifted to point a and hence the phase factor. Thus

$$\delta(t - a) = \int_{-\infty}^{\infty} e^{j2\pi f(t - a)} df$$

Let us consider an example to illustrate the method

$$g(t) = kt \qquad |t| < c$$
$$= 0 \qquad |t| > c$$

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Then as before

Now if

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$

then

$$g''(t) = \int_{-\infty}^{\infty} (j2\pi f)^2 G(f) e^{j2\pi ft} df$$

so that

$$g''(t) = -kc\delta'(t + c) + k\delta(t + c) - kc\delta'(t - c) - k\delta(t - c).$$

Thus

$$G(f) = \frac{kc(j2\pi f)\left[-e^{+j2\pi fc} - e^{-j2\pi fc}\right] + k\left[e^{j2\pi fc} - e^{-j2\pi fc}\right]}{(j2\pi f)^2}$$

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This can be simplified to

$$G(f) = -2kc^3 \left(\frac{\cos 2\pi fc}{j2\pi fc} \right) + \frac{2jk \sin 2\pi fc}{(j2\pi f)^2}$$

$$= 2jkc^2 \frac{\cos 2\pi fc}{2\pi fc} - j \frac{2kc}{2\pi f} \left(\frac{\sin 2\pi fc}{2\pi fc} \right)$$

or

$$g(t) = \int_{-\infty}^{\infty} 2jkc \left[c \frac{\cos 2\pi fc}{2\pi fc} - \frac{1}{2\pi f} \frac{\sin 2\pi fc}{2\pi fc} \right] e^{j2\pi ft} \, df$$

Generalization of the Fourier Technique

The delta function allows the generalization of the Fourier Integral technique to include the Fourier series in the case $g(t)$ contains a periodic function. This is possible because the delta function allows a Fourier Integral of complex exponential functions.

If

$$g_1(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T}} = \sum_k c_k e^{j2\pi k f_0 t} \qquad , \quad f_0 = \frac{1}{T}$$

then one has

$$G_1(f) = \int_{-\infty}^{\infty} \left[\sum_k c_k e^{j2\pi k f_0} \right] e^{-j2\pi ft} \, dt$$

$$= \sum_k c_k \int_{-\infty}^{\infty} e^{-j2\pi t [f - kf_0]} \, dt.$$

But

$$\delta(f - kf_0) = \int_{-\infty}^{\infty} e^{-j2\pi t [f - kf_0]} \, dt$$

so that

$$G_1(f) = \sum_{k=-\infty}^{\infty} c_k \, \delta(f - kf_0).$$

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As usual

$$c_k = f_o \int_t^{t + \frac{1}{f_o}} g_1(\xi) e^{j2\pi k f_o \xi} d\xi.$$

Let us take an example to show the method. First, in general

$$g(t) = g_1(t) + g_2(t)$$

where $g_1(t)$ is periodic with period T and $g_2(t)$ is non-periodic. Then the spectral resolution of $g(t)$ will be

$$G(f) = G_1(f) + G_2(f)$$

$$= \sum_{k = -\infty}^{\infty} c_k \delta(f - kf_o) + G_2(f)$$

$$= \sum_{k = -\infty}^{\infty} \{\delta(f - kf_o) \cdot f_o \int_t^{t + \frac{1}{f_o}} g_1(\xi) e^{-j2\pi k f_o \xi} d\xi \} + \int_{-\infty}^{\infty} g_2(\xi) e^{-j2\pi f \xi} d\xi$$
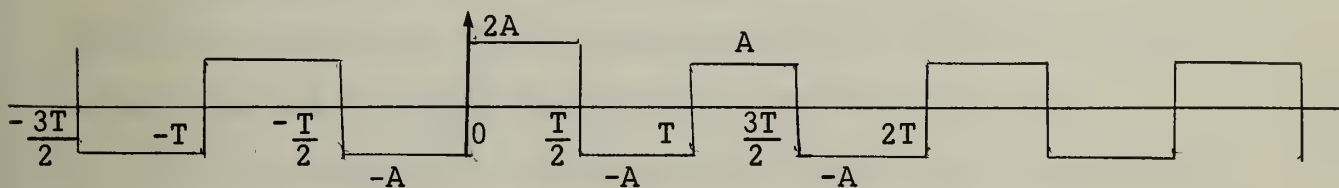
$$= \sum_{k = -\infty}^{\infty} c_k(k, f_o) \delta(f - kf_o) + G_2(f)$$

Suppose we take the case where

$$g_1(t) = A \qquad 0 < t < \frac{T}{2}$$
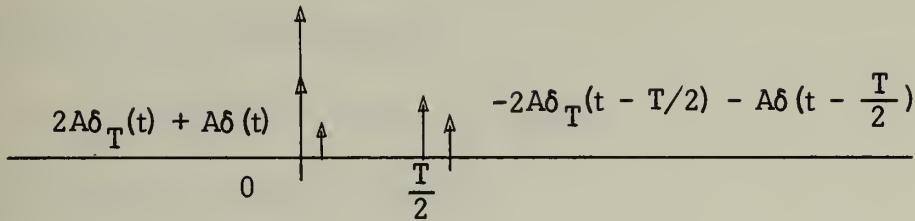$$\qquad = -A \qquad \frac{T}{2} < t < T$$

$$g_1(t) = g_1(t + T) \, , \text{ all } t$$

g(t)

We put a T subscript on the periodic δ functions so that confusion will

not result. Then as usual

$$g'(t)$$

$$2A\delta_T(t) + A\delta(t) \qquad \qquad -2A\delta_T(t - T/2) - A\delta\left(t - \frac{T}{2}\right)$$

$$0 \qquad \qquad \frac{T}{2}$$

$$c_k = \frac{2Af_0 - 2Af_0 e^{-j2\pi k f_0 \cdot \frac{1}{2f_0}}}{j2\pi k f_0} \quad , \quad G_2(f) = \frac{A - Ae^{j2\pi f \cdot \frac{1}{2f_0}}}{(j2\pi f)}$$

so that

$$G(f) = \sum_{k=-\infty}^{\infty} \frac{2A[1 - e^{-j\pi k}]}{j2\pi k} \delta(f - kf_0) + \frac{A[1 - e^{j\pi f/f_0}]}{j2\pi f}$$

and thus

$$g(t) = \int_{-\infty}^{\infty} \left[\sum_k \frac{2A(1 - e^{-j\pi k})}{j2\pi k} \delta(f - kf_0) + \frac{A(1 - e^{j\pi f/f_0})}{j2\pi f} \right] e^{j2\pi ft} \, df$$

Inverses of various Fourier Transforms:

Symbolic functions allow many inverse Fourier transforms to be

obtained by inspections. This is because many time functions of interest

can be generated by superposition of various symbolic functions.

The first fundamental symbolic function of use is the delta function,

after this we need the sgn(t) function. By consulting a table of integrals

one finds

$$\int_0^\infty \frac{\sin 2\pi ft}{2\pi f} \, d(2\pi f) = \frac{\pi}{2} \quad , \quad t > 0$$

$$= 0 \quad , \quad t = 0$$

$$= -\frac{\pi}{2} \quad , \; t < 0$$

We may write this symbolically as

$$\int_0^\infty \frac{\sin 2\pi ft}{2\pi f} \, d(2\pi f) = \frac{\pi}{2} \, \text{sgn}(t)$$

Now $\sin 2\pi ft / 2\pi f$ is even in f so that

$$\int_{-\infty}^\infty \frac{\sin 2\pi ft}{2\pi f} \, d(2\pi f) = \pi \, \text{sgn}(t)$$

We wish to express our result in exponential form so that we have

$$\frac{1}{2} \, \text{sgn}(t) = \int_{-\infty}^\infty \frac{\sin 2\pi ft}{2\pi f} \, df$$

$$= \int_{-\infty}^\infty \frac{e^{j2\pi ft} - e^{-j2\pi ft}}{(2j) \, 2\pi f} \, df$$

$$= \int_{-\infty}^\infty \left[\frac{e^{j2\pi ft}}{2j(2\pi f)} + \frac{e^{+j2\pi ft}}{2\pi f(2j)} \right] df$$

$$= \int_{-\infty}^\infty \frac{e^{j2\pi ft}}{j2\pi f} \, df$$

To obtain a representation for $Us(t)$ we note that

$$Us(t) = \frac{1}{2} + \frac{1}{2} \, \text{sgn}(t)$$

so that

$$Us(t) = \frac{1}{2} + \int_{-\infty}^\infty \frac{e^{j2\pi ft}}{j2\pi f} \, df .$$

But

$$\frac{1}{2} = \int_{-\infty}^\infty \frac{\delta(f)}{2} \, e^{j2\pi ft} \, df$$

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so that

$$Us(t) = \int_{-\infty}^{\infty} \left[\frac{\delta(f)}{2} + \frac{1}{j2\pi f} \right] e^{j2\pi ft} df.$$

If, instead we want a Heaviside function to start at a we then have

$$Us(t - a) = \int_{-\infty}^{\infty} \left[\frac{\delta(f)}{2} + \frac{1}{j2\pi f} \right] e^{j2\pi f(t - a)} df$$

From this result we easily obtain

$$sq_T(t) = Us(t) - Us(t - T)$$

$$= \int_{-\infty}^{\infty} \left[\frac{\delta(f)}{2} + \frac{1}{j2\pi f} \right] \left[1 - e^{-j2\pi fT} \right] e^{j2\pi ft} df$$

Again, from the integral tables we have

$$\int_{0}^{\infty} \frac{\cos mx}{1 + x^2} dx = \frac{\pi}{2} e^{-m} \quad , \ m > 0$$

$$= \frac{\pi}{2} \quad , \ m = 0$$

$$= +\frac{\pi}{2} e^{+m} \quad , \ m < 0$$

or

$$= \frac{\pi}{2} e^{-|m|}$$

Similarly to the above steps for $Us(t)$

$$cv_\sigma t = \frac{2}{\pi} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos \frac{2\pi f}{\sigma} \cdot \sigma t}{1 + (\frac{2\pi f}{\sigma})^2} d(\frac{2\pi f}{\sigma})$$

$$= \int_{-\infty}^{\infty} \frac{\sigma(e^{j2\pi ft} + e^{-j2\pi ft})}{\sigma^2 + 4\pi^2 f^2} df$$

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$$= 2 \int_{-\infty}^{\infty} \frac{\sigma \, e^{j2\pi ft}}{\sigma^2 + (2\pi f)^2} \, df$$

$$= \int_{-\infty}^{\infty} \frac{2\sigma \, e^{j2\pi ft}}{\sigma^2 - (j2\pi f)^2} \, df$$

The results are summarized in the table below:

$g(t)$	$F(g) = G(f)$
$\delta(t-a)$	$e^{-j2\pi fa}$
$Us(t-a)$	$\left[\dfrac{\delta(f)}{2} + \dfrac{1}{j2\pi f} \right] e^{-j2\pi fa}$
$sgn(t-a)$	$\dfrac{2}{j2\pi f} \, e^{-j2\pi fa}$
$sq_T(t-a)$	$\left[\dfrac{\delta(f)}{2} + \dfrac{1}{j2\pi f} \right] \left[1 - e^{-j2\pi fa} \right]$
$cv_\sigma^i(t-a)$	$\dfrac{2\sigma}{\sigma - (j2\pi f)^2} \, e^{-j2\pi fa}$

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